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Pendubot: Combining of Energy and Intuitive Approaches to Swing up, Stabilization in Erected Pose

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Abstract The objective of this paper is to define a strategy for the swing up of a double pendulum and its stabilization in the unstable equilibrium state with both erected links. This double-link pendulum, usually called pendubot, is an underactuated system because it has only one motor, which actuates the suspension joint. The limits on the torque amplitude are taken into account. Simulation results demonstrate that our strategy is efficient.

Keywords

Underactuated system, Pendubot, Swing up, Unstable equilibrium, Stabilization, Saturated control.

1 Introduction

Motivation. The pendulums with more joints than actuators are mechanical systems, which belong to the family of underactuated systems. They represent a difficult challenge for the control theory. The inverted pendulums can play the role of a didactic platform, see for example [1], where usual digital controls are illustrated with an inverted pendulum on a cart, the so-called Cart-Pole System. The pendulum systems attract also the attention of researchers in control as a benchmark for testing and evaluating of control strategies, to mention a little set of references only, see [2–13].

For the pendulums usually two problems have to be taken into account, which are the swing up to reach an upright unstable equilibrium and the stabilization in this position. For example with an Acrobot, which is a two-link pendulum, whose first link is not actuated whereas the second one is actuated, authors of [2] and [14] define a method in two stages. At

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first, a slow swinging up process, founded on a passivity-based approach, brings the acrobot close to the upright position. After a switching control to a balancing linear controller, the pendulum is locally asymptotically stabilized around its upright equilibrium position. A new control law to swing up an Acrobot and to stabilize it in the upright position is developed in [15]. In [16] and [17], the underactuated planar revolute robot, usually called pendubot, is presented. A pendubot is also a two-link pendulum with an actuator at the shoulder, but no actuator at the elbow. One very interesting distinction of the pendubot over both the classical cart-pole system and Furuta's system is the continuum of equilibrium positions. The partial feedback linearization technique is used to design the control that swings the two links from their hanging stable equilibrium to the unstable erected position. A linear state feedback is designed to balance the pendulum at the erected equilibrium. The LQR or pole placement technique can be used to find the feedback gains. In [5], an algorithm has been proposed to swing up a pendubot. This algorithm brings the pendulum close to the top unstable equilibrium. The second link remains swinging while getting closer and closer to the top equilibrium. In [8] the swing up controller switches to a hybrid controller for feedback stabilization among the erected position of an experimental Pendubot. This hybrid controller is composed of a continuous-time control part, which contributes for partial feedback linearization, and a discret-time control part, which can be regarded as cancelation of the drift terms. A pendubot is also considered in [18]. A variable structure controller from [19], based on the second order sliding mode method, drives the pendubot to a periodic reference orbit in finite time. A modified Van der Pol oscillator is involved into the controller synthesis as a reference model. The resulting closed-loop system is capable of moving from one orbit to another by changing the parameters of the Van der Pol oscillator. In [20], authors investigate some properties of the simple strategies for swinging up a one-link pendulum based on energy approach. The position and the velocity of the pivot are not considered. The global behavior of the swing up is completely characterized by the ratio of the maximum acceleration of the pivot and the acceleration of gravity. The swing up and the stabilization problems for the considered one-link pendulum with a limited actuator, are simultaneously solved with a single, smooth law by authors. The idea is to shape the potential energy and to introduce in the control law an additional dumping or pumping term following the state of the pendulum. In [21], the problem of the stabilization a one-link inverted pendulum around its homoclinic orbit is addressed. A control strategy, based on an energy approach of the cart and pendulum system is proposed to balance the inverted pendulum and raise it to its upper equilibrium position while the cart displacement is brought to zero. A one-link pendulum with an inertia-wheel is considered in [22]. Taking into account the limits of the actuator, authors have designed a swing up control law. The switching time between the swing up control and the stabilization control is defined by comparing the global energy of the pendulum to its potential energy in the upright equilibrium and getting its basin of attraction. Using the Jordan form of the equations of motion, the authors extract the unstable mode. Suppressing this unstable mode, they obtain a basin of attraction, which is as large as possible. The experimental results show the remarkable efficient of the design control. Lai *et al.* [13] propose a unified treatments of the motion control of underactuated two-link manipulators, including acrobots and pendubots. The global stability of the control system is analyzed and guaranteed by using arguments from the Lyapunov's theory. In [23], a stabilization control of a two-link inverted pendulum with an inertia-wheel is designed for the three unstable equilibriums. An approach of the nonlinear control design is proposed in [24] for underactuated mechanical systems and results of global stabilization for an acrobot, a Cart-pole system, an inertia-wheel pendulum, a rotating pendulum are presented. This approach is based on an explicit change of coordinates and control that transform several classes of underactuated mechanical systems into cascade nonlinear systems with structural properties that are convenient for the control design purposes.

In literature for the swing up and the local stabilization of a pendubot, to our best knowledge, first, there is no explicit way to avoid antagonistic movements of the swing links during the process of swinging up. Secondly, during the process of stabilization, it does not exist a control law with saturation, founded on the basin of attraction (or its approximation, the largest as possible) and treating explicitly the unstable modes.

Contribution. The paper proposes a control strategy that makes a pendubot upright by swinging up, and stabilizing the erected position. The limits on the torque amplitude in the suspension point are taken into account. The original switching control scheme consists of three parts:

– When the pendulum is in some neighborhood of the downward resting position, a local controller, based on the energy boosting algorithm, is employed.
– When the pendulum is out of this neighborhood, a saturated nonlinear controller is used in order to straighten the double-link pendulum (to make it close to a one-link pendulum).
– When the pendulum reaches the basin of attraction of the upright position, a saturated linear feedback is used.

The novelty in our control strategy is the following. We take into account the limits imposed on the control torque. In the process of swinging up, the pendubot with the designed control performs a number of vibrations from side to side with increasing amplitude as a one-link pendulum. Remind, under algorithm designed in [5], the first link converges to the top position after several initial oscillations, while the second link performs oscillations with increasing amplitude. Second, the gains of the saturated balancing control are chosen to ensure the basin of attraction as large as possible. The swing up control switches to the balancing mode at the instant when the system comes to the basin of attraction.

Structure of the paper. Section 2 is devoted to the model of the double pendulum. The linearized model is recalled in Section 3. The statement of the problem is defined in Section 4. Section 5 presents the controls for the local stabilization of the double-link pendulum in the unstable equilibrium posture. Section 6 is devoted to the definition of the swing up strategy. Simulation results are presented in Section 7. Finally, Section 8 presents our conclusion and perspectives.

2 Model Description of the Double-Link Pendulum

Let us consider a mechanical system with two rigid bodies depicted in Figure 1. The DC motor actuates the suspension joint O_1. But there is no actuator in the inter-link joint O_2. Let C_1 and C_2 be the centers of mass of the first and second link respectively. The center of mass C_1 is located on line O_1O_2. Let the following lengths be $O_1O_2 = l$, $O_1C_1 = r_1$ and $O_2C_2 = r_2$. Let m_1 and m_2 be the masses of the first and second links. The moment of inertia of the first link about joint O_1 is denoted I_1, the moment of inertia of the second link about joint O_2 is denoted I_2.

The generalized coordinates are the angles φ and γ, Figure 1. The joint variable $\alpha = \gamma - \varphi$ is also used in our study.

So, our system has two degrees of freedom, but one actuator only; this system is underactuated with a degree of underactuation, which equals one. Consider the following constraint

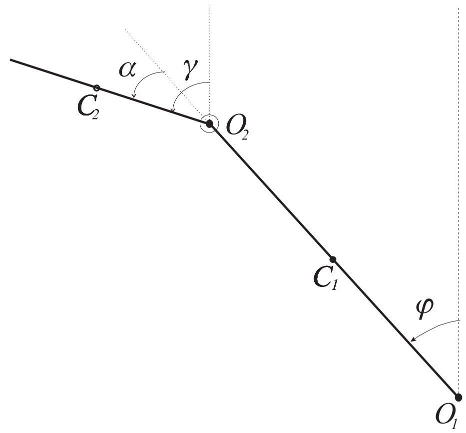

Fig. 1: Scheme of the double-link pendulum.

imposed on torque Γ, which is developed by the motor in the suspension point O_1:

$$|\Gamma| \leq \Gamma_0,\ \Gamma_0 = const \tag{1}$$

The expressions for the kinetic energy T and potential energy Π of the two-link pendulum are well-known:

$$2T = a_{11}\dot{\varphi}^2 + 2a_{21}cos(\gamma - \varphi)\dot{\gamma}\dot{\varphi} + a_{22}\dot{\gamma}^2,$$
$$\Pi = b_1 cos\varphi + b_2 cos\gamma$$

Here $a_{11} = I_1 + m_2 l^2$, $a_{21} = m_2 r_2 l$, $a_{22} = I_2$, $b_1 = (m_1 r_1 + m_2 l)g$, $b_2 = m_2 r_2 g$ (g is the gravity acceleration).

Lagrangian $L = T - \Pi$ yields the following well known equations of motion:

$$a_{11}\ddot{\varphi} + a_{21}cos(\gamma - \varphi)\ddot{\gamma} - a_{21}sin(\gamma - \varphi)\dot{\gamma}^2 - b_1 sin\varphi = \Gamma \tag{2}$$

$$a_{21}cos(\gamma - \varphi)\ddot{\varphi} + a_{22}\ddot{\gamma} + a_{21}sin(\gamma - \varphi)\dot{\varphi}^2 - b_2 sin\gamma = 0 \tag{3}$$

The angular momentum K with respect to the suspension joint O_1 of the double-link pendulum is

$$K = \frac{\partial T}{\partial \dot{\varphi}} = a_{11}\dot{\varphi} + a_{21}cos(\gamma - \varphi)\dot{\gamma} \tag{4}$$

System (2), (3) with $\Gamma = 0$ has the unstable equilibrium state:

$$\varphi_e \equiv 0,\ 2\pi, \quad \gamma_e \equiv 0,\ 2\pi \tag{5}$$

We consider the problem to transfer the double pendulum to the equilibrium posture (5) from the stable equilibrium posture

$$\varphi \equiv \pi, \quad \gamma \equiv \pi \tag{6}$$

3 Linearized Model of the Double-Link Pendulum

In the last phase of the pendulum transferring process to the desired final position, we stabilize it. The linear model, but with limits (1) imposed on the control torque is used to design the feedback for the stabilization. Let us denote the state vector by

$$\begin{aligned} x &= \left(\varphi - \varphi_e,\ \gamma - \gamma_e,\ \dot{\varphi},\ \dot{\gamma}\right)^* \\ &= \left(\varphi,\ \gamma,\ \dot{\varphi},\ \dot{\gamma}\right)^*. \end{aligned}$$

Star * means transposition. Equations (2), (3) linearized near the equilibrium (5) have the following matrix form:

$$\begin{aligned} \dot{x} &= Ax + p\Gamma \\ &= \begin{bmatrix} 0_{2\times2} & I_{2\times2} \\ D^{-1}E & 0_{2\times2} \end{bmatrix} x + \begin{bmatrix} 0_{2\times1} \\ D^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{bmatrix} \Gamma \end{aligned} \tag{7}$$

Matrices D and E are:

$$D = \begin{pmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{pmatrix}, \quad E = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}. \tag{8}$$

The determinant of the controllability matrix [25] of the model (7), (8) is

$$-\frac{b_2^2 a_{21}^2}{(a_{11}a_{22} - a_{21}^2)^4} \tag{9}$$

Value $a_{11}a_{22} - a_{21}^2 \neq 0$ because it is the determinant of the inertia matrix with $\alpha = 0$. Thus, the linear model is Kalman controllable, if and only if $r_2 \neq 0$ and $l \neq 0$.

Introducing a nondegenerate linear transformation $x = Sy$ with a constant matrix S, it is possible to obtain the well-known Jordan form of the matrix equation (7)

$$\dot{y} = \Lambda y + d\Gamma \tag{10}$$

where Λ is a diagonal matrix

$$\begin{aligned} \Lambda &= S^{-1}AS = \begin{pmatrix} \lambda_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \lambda_4 \end{pmatrix}, \\ d &= S^{-1}p = [d_1,\ d_2,\ d_3,\ d_4]^*. \end{aligned} \tag{11}$$

Here, $\lambda_1, \ldots, \lambda_4$ are the eigenvalues of the matrix A. They are the roots of the characteristic equation for the linear system (7) with $\Gamma = 0$. This characteristic equation is biquadratic

$$a_0\lambda^4 + a_1\lambda^2 + a_2 = 0 \tag{12}$$

because the system (2), (3) is conservative.

4 Statement of the Problem

The upright equilibrium posture (5) is unstable. The objective is to design a feedback control, satisfying the constraint (1), to swing up the pendulum and to stabilize it in this unstable equilibrium posture.

We consider the problem of local stabilization of equilibrium (5) firstly, whereas the stabilization is the last phase of the transferring pendulum process. The asymptotic local stabilization around the unstable equilibrium is realized here with an approach defined in [26].

5 Local Stabilization of the Double-Link Pendulum

The coefficients of the characteristic equation (12) for the system (7), (8) are:

$$a_0 = a_{11}a_{22} - a_{12}^2 = detD > 0,$$
$$a_1 = -(a_{11}b_2 + a_{22}b_1) < 0,$$
$$a_2 = b_1 b_2 = detE > 0$$

The leading coefficient a_0 is positive because it is the determinant of the positive definite matrix D; $a_1 < 0$ because a_{11}, a_{22}, b_1 and b_2 are positive values. Consequently equation (12) has two real positive roots λ_1, λ_2 and two real negative roots $\lambda_3 = -\lambda_1$, $\lambda_4 = -\lambda_2$.

We intend to design an admissible (satisfying the inequality (1)) feedback control $\Gamma(x)$ to ensure the asymptotic stability of the equilibrium state $x = 0$ of system (7) or (10) with the largest as possible basin of attraction (the larger the basin of attraction, the more robust the control).

Let us consider the first two scalar differential equations of the system (10), (11), corresponding to the positive eigenvalues λ_1 and λ_2:

$$\dot{y}_1 = \lambda_1 y_1 + d_1\Gamma, \quad \dot{y}_2 = \lambda_2 y_2 + d_2\Gamma \tag{13}$$

The system (7), (8) is Kalman controllable. Therefore, the subsystem (13) is also controllable (see [25]) and $d_1 \neq 0$, $d_2 \neq 0$.

Let W be the set of the piecewise continuous functions $\Gamma(t)$, satisfying the inequality (1). Let Q be the set of the initial states $x(0)$ of the system (7), from which origin $x = 0$ can be reached, using an admissible control functions $\Gamma(t) \in W$. In other words, the system (7) can reach the origin $x = 0$ with control $\Gamma(t) \in W$, only starting from the initial states $x(0) \in Q$. Set Q is called controllability domain. If matrix A has eigenvalues with positive real parts and the control variable Γ is restricted, then the controllability domain Q for the system (7) is an open subset of the phase space X (see [27]).

For any admissible feedback control $\Gamma = \Gamma(x)$ (with the saturation $|\Gamma(x)| \leq \Gamma_0$) the corresponding basin of attraction B belongs to the controllability domain: $B \subset Q$. Here, as usual, B is the set of the initial states $x(0)$, from which the system (7), with the feedback $\Gamma = \Gamma(x)$ asymptotically tends to the origin $x = 0$ as $t \to \infty$.

The controllability domain Q' of the system (13) in plane y_1, y_2 is an open bounded set with the following boundaries (see [27], [28])

$$\begin{aligned} y_1(\tau) &= \pm\frac{d_1\Gamma_0}{\lambda_1}\left(2e^{-\lambda_1\tau} - 1\right), \\ y_2(\tau) &= \pm\frac{d_2\Gamma_0}{\lambda_2}\left(2e^{-\lambda_2\tau} - 1\right) \quad (0 \leq \tau < \infty) \end{aligned} \tag{14}$$

The boundary of the controllability region Q' has two corner points depicted in Figure 2:

$$
\begin{gathered}
y_1 = -d_1 \frac{\Gamma_0}{\lambda_1}, \quad y_2 = -d_2 \frac{\Gamma_0}{\lambda_2}; \\
y_1 = d_1 \frac{\Gamma_0}{\lambda_1}, \quad y_2 = d_2 \frac{\Gamma_0}{\lambda_2}
\end{gathered}
\tag{15}
$$

These points (15) are the equilibrium points of the system (13) under the constant controls:

$$
\Gamma = \pm\Gamma_0 \tag{16}
$$

We can "suppress" the instability of equilibrium $y_1 = 0$, $y_2 = 0$ of the system (13) by a linear feedback control,

$$
\Gamma = k_1 y_1 + k_2 y_2 \tag{17}
$$

with $k_1, k_2 = const$. It is shown in the paper [29] and the book [30] that using a linear feedback (17) with saturation ($\beta = const$),

$$
\Gamma = \begin{cases} \Gamma_0, & if\ \beta(k_1 y_1 + k_2 y_2) \geq \Gamma_0 \\ \beta(k_1 y_1 + k_2 y_2), & if\ |\beta(k_1 y_1 + k_2 y_2)| \leq \Gamma_0 \\ -\Gamma_0, & if\ \beta(k_1 y_1 + k_2 y_2) \leq -\Gamma_0 \end{cases} \tag{18}
$$

the basin of attraction B' of the system (13), (18) can be made arbitrary close to the controllability domain Q'.

The straight line crossing two points (15) is the following:

$$
k_1 y_1 + k_2 y_2 = 0
$$

with

$$
k_1 = -\frac{d_2}{\lambda_2}, \quad k_2 = \frac{d_1}{\lambda_1} \tag{19}
$$

If

$$
sign\beta = sign\left[d_1 d_2 \left(\lambda_1 - \lambda_2\right)\right]
$$

and $|\beta| \to \infty$, then the basin of attraction B' of the system (13) under the nonlinear control (18) with the coefficients (19) tends to the controllability region Q'. Consequently, using the gains (19), basin B' (two-dimensional) can be made arbitrary close to domain Q'. If $|\beta| \to \infty$, the control (18) tends to the bang-bang control.

Solutions $y_1(t)$ and $y_2(t)$ of equations (13) with control (18) tend to 0 as $t \to \infty$ for the initial values $y_1(0)$, $y_2(0)$, belonging to the basin of attraction B'. But if $y_1(t) \to 0$ and $y_2(t) \to 0$, then, according to the expression (18), $\Gamma(t) \to 0$. Therefore, solutions $y_3(t)$, $y_4(t)$ of the third and fourth equations of system (10) with any initial conditions $y_3(0)$, $y_4(0)$ converge to zero as $t \to \infty$, because $\lambda_3, \lambda_4 < 0$. Thus, under control (18) with coefficients (19), the basin of attraction B (four-dimensional) of system (10), (18) is described by the same relations, which describe the basin of attraction B' (two-dimensional) of system (13), (18). Basin B (and controllability domain Q) is limited in unstable coordinates y_1, y_2 only. The boundary of the basin B' is the periodical cycle of the system (13), (18). This cycle can be computed, using the backward motion of the system (13), (18) from a state close to the origin $y_1 = y_2 = 0$.

Variables y_1 and y_2 depend on the original variables from the vector x, according to transformation $y = S^{-1}x$. Due to this, formula (18) defines a nonlinear feedback control, which depends on vector x of the original variables.

According to Lyapounov's theorem (see [31]), the equilibrium state $x = 0$ of the nonlinear system (2), (3) is asymptotically stable under the control (18) because system (2), (3), (18) linearized near the state $x = 0$ is asymptotically stable.

The double-link pendulum is here assumed with similar homogeneous links and parameters:

$$m_1 = m_2 = 0.2\,kg,\ l = 0.15\,m,\ \Gamma_0 = 0.2\,N{\cdot}m \tag{20}$$

With parameters (20) the positive eigenvalues λ_1, λ_2 are the following: $\lambda_1 = 18.5611, \quad \lambda_2 = 6.920$.

Using formulas (14) we design the controllability domain Q' depicted in Figure 2. The boundary of this domain is shown in Figure 2 by dashed line; the boundary of the basin of attraction B' with $\beta = 0.025$ is shown by solid line. If $\beta = 0.1$, then basin B' is very close to domain Q' and it is difficult to see on the sketch the difference between them. In this case, we can use domain Q' as a basin of attraction.

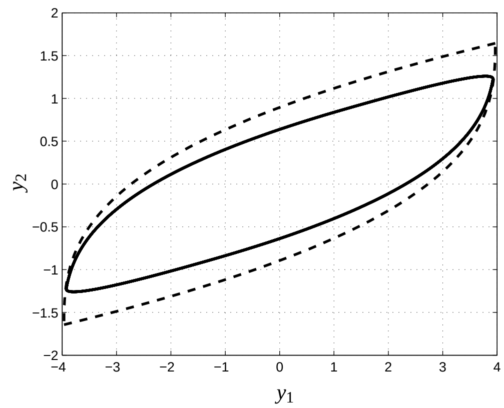

Fig. 2: The boundaries of the controllability domain Q' (dashed line) and of the basin of attraction B' (solid line) with $\beta = 0.025$.

6 Swing up Via Energy and An Intuitive Control

In this Section, we describe two stages of the developed control. The process of the local stabilization starts after these two stages.

6.1 Energy approach

The process of the double-link pendulum transferring from the stable equilibrium (6) to the unstable one (5) contains several phases. First, the pendulum is swinging up to increase its total mechanical energy $T + \Pi$. This energy in the desired unstable equilibrium state, which is the upright position (5) equals to the potential energy $P = b_1 + b_2$.

In the area

$$\pi - \Delta\varphi \le \varphi \le \pi + \Delta\varphi \tag{21}$$

the following control

$$\Gamma = \begin{cases} \Gamma_0/\sigma, & if\ \dot{\varphi} > 0 \\ -\Gamma_0/\sigma, & if\ \dot{\varphi} \le 0 \end{cases} \tag{22}$$

is used. Here $\Delta\varphi = const > 0$, $\sigma > 0$ are parameters, which we choose by simulation. Under the control (22) (in area (21)), the total energy $T + \Pi$ increases monotonically because its time derivative changes according to the equality:

$$\frac{d(T+\Pi)}{dt} = \Gamma\dot{\varphi} = |\dot{\varphi}|\Gamma_0/\sigma \tag{23}$$

6.2 Double pendulum straightening

Out of area (21) we try to keep angle α near zero, because $\alpha = 0$ in the desired upward equilibrium state (5). Also straightening the double pendulum, we try to avoid the movements of the links in the opposite directions. If angle α is close to zero, then the double-link pendulum is "similar" to a one-link pendulum and it is naturally to swing up a one-link pendulum with the control (22).

From system (2), (3) to make the exact partial input-output feedback linearization such that

$$\ddot{\alpha} = -c_2\dot{\alpha} - c_1\alpha, \tag{24}$$

the control torque is:

$$\begin{aligned} \Gamma = \Gamma_d = &-a_{21}(\dot{\varphi}+\dot{\alpha})^2 sin\alpha - b_1 sin\varphi + \\ &\frac{(c_1\alpha + c_2\dot{\alpha})(a_{22}a_{11} - a_{21}^2 cos^2\alpha)}{a_{21}cos\alpha + a_{22}} + \\ &\frac{(a_{11} + a_{21}cos\alpha)[b_2 sin(\varphi+\alpha) - a_{21}\dot{\varphi}^2 sin\alpha]}{a_{21}cos\alpha + a_{22}} \end{aligned} \tag{25}$$

Here c_1 and c_2 are positive feedback gains. The denominator in expression (25) is far from zero, if angle α is close to zero. Under control (25), system (2), (3) has (independently of the behavior of angle φ) an asymptotically stable solution:

$$\alpha(t) = 0 \quad (\varphi(t) = \gamma(t)) \tag{26}$$

If the inequality (1) is taken into account, the saturated control law can be defined instead of (25):

$$\Gamma = \begin{cases} \Gamma_0, & if\ \Gamma_d \ge \Gamma_0 \\ \Gamma_d, & if\ |\Gamma_d| \le \Gamma_0 \\ -\Gamma_0, & if\ \Gamma_d \le -\Gamma_0 \end{cases} \tag{27}$$

Let us note:

- If we put $\alpha = \dot{\alpha} = 0$ in formula (25), then we obtain the following expression:

$$\Gamma = \left(\frac{a_{11}+a_{21}}{a_{21}+a_{22}} b_2 - b_1 \right) sin\varphi$$

Under this control torque, the system (2), (3) has solution (26).
The simulation demonstrates that in order to straightening the pendulum it is also possible to use a simpler control than (25):

$$\Gamma = \left(\frac{a_{11}+a_{21}}{a_{21}+a_{22}} b_2 - b_1 \right) sin\varphi + c_1\alpha + c_2\dot{\alpha}$$

- Consider system

$$\ddot{\alpha} = v, \quad |v| \leq v_0, \quad v_0 = const \tag{28}$$

The time-optimal control $v(\alpha, \dot{\alpha})$, which brings system (28) to origin $\alpha = \dot{\alpha} = 0$ is [28, 32]:

$$v = -v_0 sign(2v_0\alpha + \dot{\alpha}|\dot{\alpha}|) \tag{29}$$

Instead of the discontinuous function (29) it is better for the realization to use the following continuous function

$$v = -v_0 th(\tau[2v_0\alpha + \dot{\alpha}|\dot{\alpha}|\,]) \tag{30}$$

with τ as parameter. Substituting expression (30) instead of $-c_1\alpha - c_2\dot{\alpha}$ in formula (25) we obtain

$$\Gamma = \Gamma_d = -a_{21}(\dot{\varphi}+\dot{\alpha})^2 sin\alpha - b_1 sin\varphi + \frac{v_0 th(\tau[2v_0\alpha + \dot{\alpha}|\dot{\alpha}|\,])(a_{22}a_{11} - a_{21}^2 cos^2\alpha)}{a_{21}cos\alpha + a_{22}} + \frac{(a_{11}+a_{21}cos\alpha)[b_2 sin(\varphi+\alpha) - a_{21}\dot{\varphi}^2 sin\alpha]}{a_{21}cos\alpha + a_{22}} \tag{31}$$

We also tested sucessfully this control (31) with several parameters v_0 and τ on the stages of the pendulum straightening.

Each stage of our control strategy is provided by mathematical consideration. But the demonstration of the efficiency of the complete control strategy is supported by successful computer simulation and intuition only (see below).

7 Simulation

The double-link pendulum is simulated with the numerical parameters (20), and we have chosen $\Delta\varphi = \pi/12$. But the control law is not very sensitive to the value $\Delta\varphi$; the values $\Delta\varphi$ in the interval $[\pi/24,\ \pi/6]$ are also acceptable. If the value $\Delta\varphi$ decreases, the time of the swing up increases.

The appropriate coefficients $c_1 = 12\ rad^{-1}$ and $c_2 = 1.2\ s.rad^{-1}$ of the feedback control (27) are chosen in the simulation. The gains c_1, c_2, which significantly differ from the chosen values can be also used successfully.

As we see in simulation under control (22), (27), the pendulum sways from side to side and its energy $T + \Pi$ increases ”in average” (not monotonically). The stages of the energy boosting (see control (22)) and the straightening of the pendulum (see control (27)) are alternating.

If value Γ_0/σ is too large, the pendulum may race through the desired equilibrium (5). In simulation the value $\sigma = 7$ is used. This means that with the control law (22) we do not use all the resources of the actuator.

After several vibrations, control (22), (27) brings the double-link pendulum close to the upright position (5) with angular velocities close to zero and to the basin of attraction. (Remind we try to obtain the basin of attraction as large as possible.) When the system reaches the basin of attraction, the control law (22), (27) switches on the stabilization control law (18) (with $\beta = 0.1$). At this time, the phase of the pendulum stabilization in the equilibrium (5) starts. This is the last phase of the pendubot transferring process.

To design a more robust control it is useful to increase the coefficient σ, when the full energy becomes closer to the potential energy $P = b_1 + b_2$ of the pendulum in the upright position (5).

Unlike the control law (22), all the resources of the actuator are used in the control laws (27) and (18).

In Figure 3, the process of the pendubot swinging up and its stabilization in erected pose is shown. The amplitude of angle α vibrations becomes large initially, but after it becomes small and the pendubot sways from side to side like a one-link pendulum. (Note, control (22) is natural to swing up the one-link pendulum.) The amplitude of these vibrations increases. After some number of vibrations, at time 14.7 s approximately, the system comes to the basin of attraction and after to the desired equilibrium.

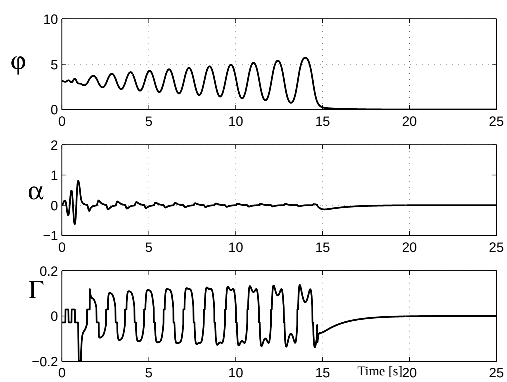

Fig. 3: Swing up with control (22), (27) and stabilization in state (5) with control (18), $\varphi \to 0$, $\alpha \to 0$, $\Gamma \to 0$.

In Figure 4, the graph of the total energy $T + \Pi$ is shown. The energy increases ”in average” and at the end becomes the desired constant $P = b_1 + b_2$.

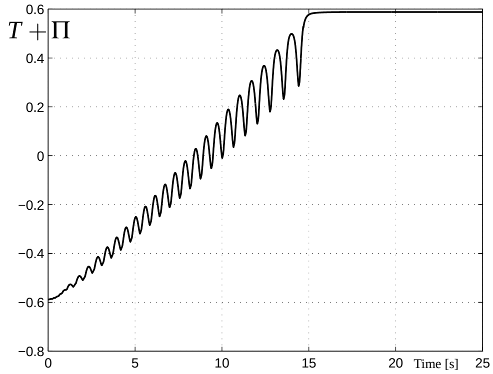

Fig. 4: Swing up with control (22), (27) and stabilization in state (5) with control (18), $T+\Pi \to b_1 + b_2$.

In area (21), instead of (22), the control law

$$\Gamma = \begin{cases} \Gamma_0/\sigma, & if\ K > 0 \\ -\Gamma_0/\sigma, & if\ K \leq 0 \end{cases} \tag{32}$$

can be also used. Here K is the angular momentum (4). In Figure 5, the corresponding process of the swinging up and stabilization is plotted. The maximal amplitude of the angle α vibrations is less than with control (22).

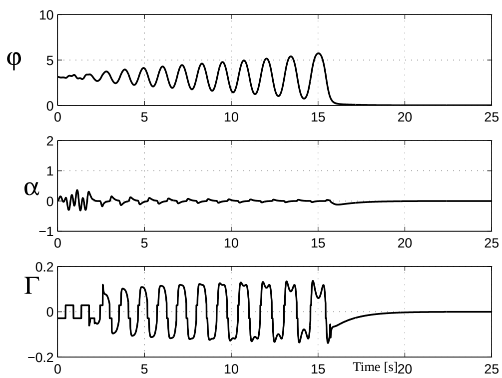

Fig. 5: Swing up with control (32), (27) and stabilization in state (5), $\varphi \to 0$ with control (18), $\alpha \to 0$, $\Gamma \to 0$.

Using the designed controls we can bring the pendubot to the upright position (5) not from the initial downward position (6) only, but also from some another initial states. For example, Figure 6 shows that it is possible to erect both links, starting from the state $\varphi = \pi$, $\gamma = \pi + \pi/3$ ($\alpha = \pi/3$), $\dot{\varphi} = \dot{\gamma} = 0$, which significantly differs from the state (6).

It is obviously that the double-link pendulum can be transferred to the stable equilibrium (6) from any initial state. Thus, the control law that transfers pendulum from the downward equilibrium (6) to the upward equilibrium (5) ensures the *global stability* of this upward equilibrium.

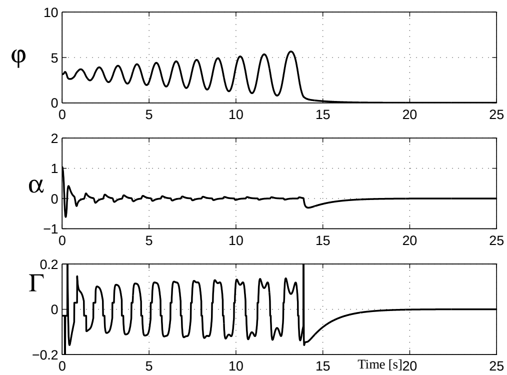

Fig. 6: $\alpha(0) = \pi/3$, Swing up with control (32), (27) and stabilization in state (5) with control (18), $\varphi \to 0$, $\alpha \to 0$, $\Gamma \to 0$.

8 Conclusion

The pendulum systems become an exciting topic in the control theory. The object of this paper is a double-link pendulum. This pendulum belongs to the family of the double-link pendulums, so-called pendubot. A theoretical and numerical study is done with simulation of its swing up and stabilization in the unstable equilibrium state with two erected links. The strategy is founded on the nonlinear control law. This control is based on the combining of the energy and intuitive approaches. The number of parameters, which are needed to find by simulation is small. Furthermore all of them have a physical sense. Under the designed feedback control, the pendubot performs a number of vibrations from side to side with increasing amplitude as a one-link pendulum. At every cycle we slightly increase the energy of the system.

We have successfully developed and tested two energy boosting algorithms and several algorithms to straighten the pendulum.

The basin of attraction is a natural criterion to define the switching time between the swing up process and the stabilization of the pendulum in the equilibrium state. The larger the basin of attraction, the more robust the control law and the shorter the duration of the erection process. Thus, it is important to have a basin of attraction as large as possible. The basins of attraction for the nonlinear model and for the linear model are close. We think that our study is of a theoretical interest and is also interesting for the education. The simulation tests demonstrate that the perspective of our swing up control and the stabilization of an experimental two-link pendulum is realistic.

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