Extension of dynamic programming models for management optimization from single structure to multi-structures level

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Extension of dynamic programming models for management optimisation from single structure to multi-structures level

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The aim of this article is the management optimisation (inspection, maintenance and rehabilitation (IM&R)) of a group of structures. It is supposed that the optimisation is constrained by limited available budget at the beginning of each time period during the entire time horizon thus creating an economical dependence between the decisions related to each of the structures. A Lagrangian relaxation technique is used for the extension of existing dynamic programming methods from single structure to multi-structures level. The methodology is illustrated by using a Generalised Partially Observable Markov Decision Process having a decision tree composed of a sequence of two decisions at the beginning of each time period, namely an inspection decision followed by a maintenance action decision. A numerical example concerning the optimisation of IM&R of 16 different bridges is presented.

Keywords: optimisation; maintenance; decision-making; infrastructure management; imperfect information; life-cycle cost

Introduction

The inspection, maintenance and rehabilitation (IM&R) management of large civil engineering infrastructure was the subject of intensive research in the past decades. Although significant progress was made, the existing methodologies still need to be improved. Such an improvement can have a substantial impact on the rational use of increasingly scarce resources for the lifetime management of civil engineering infrastructure around the world. This is because of the exponentially growing size of these structures during the last half century.

Maintenance models based on Markov Decision Processes (MDPs) are widely accepted and used within the management community for IM&R or M&R optimisation of civil engineering infrastructures. These models are especially suitable for sequential decision optimisation problems, i.e. problems in which the present decision may affect the circumstances under which future decisions will be made. Since dynamic programming was proposed by Bellman (1952) as a solution for sequential optimisation problems, it has been readily adapted to deal with maintenance optimisation problems and was extensively used in machinery maintenance optimisation. However, it had a major limitation; namely, a perfect inspection was implicitly assumed to be done at the beginning of each stage. Partially Observable Markov Decision Processes (POMDPs) (Drake 1962, Eckles 1968, Monahan 1982) are a generalisation of MDPs in which it is not assumed that the state of the system at each decision stage is precisely known. Previous research (Madanat and Ben-Akiva 1994, Corotis *et al.* 2005) introduced the dynamic programming models with partially observable states of the system for IM&R optimisation of single structures. In their approach, an optimal sequence of two decisions is prescribed during each time period. It consists of (i) choosing among a set of feasible imperfect inspection techniques, a single inspection technique to apply to the structure and (ii) choosing among a set of feasible imperfect maintenance actions, a single maintenance action to be implemented. The maintenance actions are assumed to be imperfect in the sense that their effect on the state of the structure is uncertain; the inspections are also assumed imperfect in the sense that given a specific state of the structure, their results are uncertain. In such models, the deterioration model is assumed to be uncertain, i.e. stochastic. This approach has been extended (Faddoul *et al.* 2009) to a so-called Generalised Partially Observable Markov Decision Process (GPOMDP) where any sequence of decisions of any length can be optimised during each time period.

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Regardless of the type of the model used for IM&R optimisation of single structures, its extension to the multi-structure case is far from being straightforward. This is mainly due to the interdependencies that may exist among the structures within the group. These interdependencies can be either structural/functional, stochastic or economic (Durango-Cohen and Sarutipand 2007). Economical interdependence may be due to the fact that the cost of maintaining a subset of assets may be different from the sum of the costs of maintaining each of the assets independently (Dekker *et al*. 1997). The effect of supply and demand on the costs of resources needed for the management of bridge structures is an example of such economical interdependence (Adey *et al*. 2003). Karabakal *et al*. (1994) proposed a zero-one integer program based on a branch and bound algorithm using Lagrangian relaxation for a parallel replacement problem with economical interdependencies due to the presence of budget constraints during each of the time periods of the planning horizon. Such economical interdependence is of particular interest to infrastructures managers. This issue received particular attention in the specialised literature related to the management of civil engineering infrastructures (Gharaibeh *et al*. 2006) and is dealt with in this article. Many of the existing models used in the case of budget constraints for the different time periods were formulated as constrained MDPs using linear programming optimisation for IM&R and/or M&R of a group of structures (Golabi *et al*. 1982, Murakami and Turnquist 1985, Madanat *et al*. 2006). The optimisation variables were chosen as being the probabilities of implementing the different maintenance activities prescribed by randomised policies. These probabilities can be intuitively interpreted as being the fraction of facilities receiving a specific action, in the context of multiple structures. This methodology was adopted to avoid computational intractability hindering multidimensional MDPs where the number of possible states of the system increases exponentially with the number of facilities in the system. However, such an approach which is based on randomised policies, is limited to homogenous systems (with respect to costs, activities types and deterioration). In reality, an agency should solve a number of M&R optimisations, one for each homogenous group of facilities and each with its own budget constraints (Madanat *et al*. 2006). Thus, the allocation of funds across the different groups and for each of the time periods of the planning horizon is highly combinatorial and computationally intractable even for a very small number of groups.

The methodology presented herein is for optimal strategy planning of IM&R decision-making concerning a group of structures and/or infrastructure assets over a prescribed time horizon. The proposed methodology uses existing single structure MDP or POMDP methodologies using discrete-space dynamic programming for IM&R or M&R optimisation. The structures within the group do not have to be similar. Thus, different sets of feasible maintenance actions, different sets of feasible inspection techniques, different models for deterioration over time and different direct and indirect costs can be assigned to each single structure of the group. The IM&R optimisation of the group of structures is supposed to be constrained by limited available budget at the beginning of each time period during the entire time horizon. Thus, the only interdependencies that we consider are economical interdependencies due the limited available budgets. The proposed methodology is based on the use of the Lagrangian relaxation technique (Everett 1963, Fisher 1981) in order to eliminate the computational intractability created by the budget constraints.

This article focuses on the extension of GPOMDPs (Faddoul *et al*. 2009) from single structure to multiple-structure level. However, the proposed methodology can also be used to extend any single structure methodology using MDP, POMDP, Corotis *et al*. model (2005) or Madanat and Ben-Akiva model (1994). In the next section, a brief overview of GPOMDP is presented. In a later section, this model is extended to the multiple-structure level. The article ends with the presentation of a numerical example that considers IM&R optimisation management of 16 different bridges for various maximum available budgets for each time period.

GPOMDP

In a Markov Decision Process using probabilistic dynamic programming for sequential maintenance optimisation, uncertainty due to the two following factors is accounted for (Hillier and Lieberman 2005): imperfect nature of maintenance actions and stochastic deterioration of the structure during time (Raphael *et al* 2008).

It should be emphasised here that classical probabilistic dynamic programming used for maintenance models implicitly presumes perfect inspection at the beginning of each time period. Such assumption implies two main drawbacks: (i) the inspections are assumed to be perfect and this is rarely the case and (ii) the optimisation of inspection planning is not possible. POMDPs where the state of the structure is uncertain (belief state), are a generalisation of MDPs, i.e. the inspection at the beginning of each time period is imperfect. A POMDP is in fact a MDP over the belief states (Eckles 1968, Monahan 1982). Contrary to a classical POMDP where at each stage the structure manager must decide for a single action decision; in a GPOMDP, the manager has at the beginning of each time period the opportunity to 'optimally' decide for a sequence of decisions to apply to the structure during that stage. This sequence of decisions consists usually of one or several inspections and/or actions applied sequentially. The choice of an action following an inspection depends on a Bayesian update of the belief state given the inspection result. For example, a 'sequence of decisions' consisting of two inspection decisions followed by one action decision is suitable for maintenance problems where a more precise and costly inspection is implemented on the basis of the results of a relatively cheap type of inspection. It is also suitable when specialised inspections technologies in detecting some of the states of the structure are implemented on the basis of the results given by an inspection technology which is efficient over all the state space of the structure. In a GPOMDP, a decision tree is used for the recursive relation of the dynamic programming. The decision tree applied to each belief state at the beginning of each time period n can be considered as a function of two variables: The belief state v^n and the optimal costs for all the belief states at the beginning of time period $n+1$, i.e. $\{{}^*c(v^{n+1})\ (\ v^{n+1}\}$. Given these variables, the decision tree will give an optimal expected cost and an optimal sequence of decisions concerning inspection types and action types to be applied to the structure. An exhaustive enumeration of the relevant assumptions of GPOMDP can be found in Faddoul *et al.* (2009).

In a GPOMDP the 'belief state' of the system at the beginning of stage n is defined by the vector $v^n = [v_1^n, v_2^n, ..., v_K^n]$ where the v_i^n are the probabilities associated with the different states θ_i, i.e. $v^n = [\Pr(\theta_1{}^n), \Pr(\theta_2{}^n), \ldots, \Pr(\theta_K{}^n)]$. The effect of maintenance action or degradation process can be modelled by transition matrices A and M, respectively where an element a_{ij} of A represents the probability that the system evolves from the state θ_i^n to the state ${}^a\theta_j^n$ if we implement the maintenance action A at the beginning of stage n and where an element m_{ij} of M represents the probability that the system evolves from the state ${}^a\theta_i^n$ to the state θ_j^{n+1} as a result of the degradation process.

The belief state ${}^a v^n$ of the system during stage n, after the implementation of a maintenance action, will be equal to the matrix product of the vector v^n by the maintenance transition matrix A_{an}, i.e. ${}^a v^n = v^n \times A_{an}$. Similarly, the belief state v^{n+1} of the system at the beginning of stage $n+1$, that is, after the evolution of the system, due to the Markovian degradation process, will be equal to the matrix product of the vector ${}^a v^n$ by the Markovian degradation process transition matrix M, i.e. $v^{n+1} = {}^a v^n \times M$.

A general case is considered herein where the inspections are imperfect, i.e. given the true state of the structure θ^n, and the used inspection technology i^n, the inspection results r_l will be characterised by a conditional probability distribution $(\Pr(r_1|\theta^n, i^n), \Pr(r_2|\theta^n, i^n), \ldots, \Pr(r_K|\theta^n, i^n))$ which is specific to each type of inspection. These probability distributions which characterise the inspections uncertainty are among the input variables of the problem.

Given the belief state vector v^n, i.e. $\Pr[\theta_1{}^n]$, $\Pr[\theta_2{}^n], \ldots, \Pr[\theta_K{}^n]$ and, the inspection technique i^n, the probability of obtaining the different results r_l, where $l = 1 \ldots K$, can be readily obtained by:

$$\Pr[r_l] = \Pr[r_l|\theta_1^n, i^n] \times \Pr[\theta_1^n] + \Pr[r_l|\theta_2^n, i^n] \times \Pr[\theta_2^n] + \cdots + \Pr[r_l|\theta_K^n, i^n] \times \Pr[\theta_K^n]. \quad (1)$$

On the other hand, given a belief state vector v^n, an inspection technique i^n and an inspection result r_l, the Bayesian posterior belief state vector ${}'v^n$ can be obtained by calculating each of its components using Bayes formula as follows:

$$\Pr[\theta_k^n|r_l,i^n] = \frac{\Pr[r_l|\theta_k^n,i^n]\Pr[\theta_k^n]}{\Pr[r_l|\theta_1^n,i^n]\Pr[\theta_1^n]+\Pr[r_l|\theta_2^n,i^n]\Pr[\theta_2^n]+\cdots+\Pr[r_l|\theta_K^n,i^n]\Pr[\theta_K^n]} \quad (2)$$

Hence, after the implementation of the maintenance action during time period j, the probability distribution of the state ${}^a\theta^j$, i.e. ${}^a v^j$ depends on the initial belief state v^1, on the degradation process, on all the maintenance actions applied to the structure during each period since the beginning of the planning horizon, and also on all the implemented inspections during each period and their obtained results since the beginning of the planning horizon.

The total expected long-term cost C_q, for structure q (where $q = 1, \ldots, Q$), is the sum of the discounted costs $cs({}^a v^n)$ generated by the state of the system during each stage of the planning horizon in addition to the discounted costs $ca(a^n)$ of the maintenance actions and the discounted costs $\sum_{l=1}^{L} ci(i_l^n)$ of the inspections (L being the number of inspections performed at each stage), i.e.

$$C_q = \sum_{j=1}^{N} \frac{1}{(1+\alpha)^j} \left[cs\left({}^a v^j\right) + ca(a^j) + \sum_{l=1}^{L} ci(i_l^j) \right]. \quad (3)$$

If we disregard any effects due to potential interdependencies between structure q and any other structure, then solving Equation (3) for the optimal strategy ${}^*x_q$, and hence for the minimum expected cost, is assumed to be done in this article using dynamic programming for the finite horizon case. A strategy x_q is a set of decision rules which associates to each possible belief state v^n an action or an inspection to be done. If an inspection is prescribed then the strategy x_q prescribes to each of its possible results an action or another optimal inspection. One can note that the objective function (3) includes two conflicting objectives: (i) minimise the direct costs $Cr_q^n = ca(a^n) + \sum_{l=1}^{L} ci(i_l^n)$ to be paid by the inventory manager for the IM&R of structure q and (ii) minimise the costs $cs(^a v^n)$ generated by the state of the system during each stage of the planning horizon. These objectives are conflicting since minimising $cs(^a v^n)$ requires a better system state and thus more maintenance actions and inspections. If we do not take into account the costs $cs(^a v^j)$ due to the state of the structure then the trivial solution to Equation (3) will be to do nothing and to let the structure deteriorate freely. Since the expected costs C_q and Cr_q^n depend on the chosen strategy x_q we will refer to these costs in the remainder of this article respectively by $C_q(x_q)$ and $Cr_q^n(x_q)$.

Extension to the multiple-structure case

If the budgets allocated to the manager of a group of structures at the beginning of each time period were unlimited, then an obvious solution to the problem consists in applying to each structure the actions and/or inspections prescribed by the single structure methodology for that particular structure. Such a solution would obviously be the best considering that it will lead to the lowest possible total expected cost required for IM&R of all the structures in the long-term. Introducing additional constraints would lead to a new optimal solution which will have in the long-term a higher expected minimal cost than the solution of the original unconstrained problem. Such additional constraints could be the limited budgets allocated to IM&R of the network at the beginning of each time period. The decision concerning the budgets allocated to IM&R for each time period is usually dictated either (i) by wider considerations pertaining to the general strategy of the company if the considered group of structures was managed by the private sector or (ii) by budgetary, economical and political considerations in case the manager was the public sector.

Problem definition

The problem of searching of optimal decisions for several simultaneous but independent sequences of decisions over a finite time horizon and which is subjected to the constraint that the resources available for the different time periods are limited can be theoretically modelled as a multidimensional MDP. Thus, the state space of the system is essentially a Cartesian product of the state spaces of each facility. This is due to the fact that one must take into account the effect of the interaction of choices in each chain on the whole problem through the limited available resources for all the chains. Such an approach is computationally intractable even for a very small number of resource-competing structures. For example, considering the case where each chain has seven different possible states at the beginning of each time period (in POMDPs the number of possible states can be in the order of hundred of thousands), the number of possible combinations for an inventory containing 15 structures will be $7^{15} = 4.7 \times 10^{11}$ combinations. We should mention as examples the Texas State bridge inventory which contains around 50,000 bridges and the more than 40,000 sections considered by Golabi and Pereira (2003) when developing a pavement management system for a road network. Clearly, the intractable complexity of MDPs having resource constraints originates from these constraints. Therefore, a natural approach to solve the problem will be to try to find a new formulation of the same problem without the constraints being explicit.

Lagrangian relaxation technique

Let x be a global strategy belonging to the set Δ of possible strategies. In other words, x represents a specific planning of inspection and maintenance actions conditioned by the (belief) states and the inspection results for all the structures during the entire time horizon of the IM&R planning. Thus, x can be considered as a Q-tuple $x = (x_1,.., x_q,.., x_Q)$ where component x_q represents the strategy for structure q.

The total expected cost is the sum of the expected cost of the different structures, i.e. $C(x) = \sum_{q=1}^{Q} C_q(x_q)$. The expected direct cost to be paid by the inventory manager for the IM&R of all the structures during time period n is the sum of the expected direct costs of the different structures, i.e. $Cr^n(x) = \sum_{q=1}^{Q} Cr_q^n(x_q)$.

The problem of minimising the total expected cost under limited budget constraints can be formalised by the following expressions:

$$\min_{x \in \Delta} C(x)$$
$$Cr^n(x) \le bl^n \;\; n = 1, 2, ..., N \tag{4}$$

where bl^n is the available budget for the inventory manager during time period n and N is the time horizon of the planning.

The Everett theorem (Everett 1963, Fisher 1981) modified to the minimise case, states that if we are given $\lambda^n (n = 1, \ldots, N)$ nonnegative real numbers, then the solutions $*x$ of the problem:

$$\min_{x \in \Delta} \left[C(x) + \sum_{n=1}^{N} \lambda^n Cr^n(x) \right] \tag{5}$$

are also solutions of the following problem:

$$\min_{x \in \Delta} C(x) \tag{6}$$

$$Cr^n(x) \leq Cr^n(^*x) \quad \text{for} \quad n = 1, 2, ..., N$$

Evidently, one obtains different optimal solutions $*x$ depending on the used values of λ^n in Equation (5). Also, the right-hand term $Cr^n(*x)$ of the limiting constraint in Equation (6) will also be different depending on the used values of λ^n. Thus, a possible approach to solve problem (4) is to solve repeatedly problem (5) for various values of λ^n until one obtains $Cr^n(*x) \approx bl^n$ for $n = 1, \ldots, N$. In the next section, it is shown that minimising problem (5) is equivalent to minimising Q sub-problems as follows:

Proposition:

Minimising problem (5) is equivalent to minimising Q sub-problems (one sub-problem for each structure), namely:

$$\begin{aligned} &\min_{x \in \Delta} \left[C(x) + \sum_{n=1}^{N} \lambda^n Cr^n(x) \right] \Leftrightarrow \\ &\min_{x_q \in \Delta_q} \left[C_q(x_q) + \sum_{n=1}^{N} \lambda^n Cr_q^n(x_q) \right] \quad for \quad q = 1, 2, .., Q \end{aligned} \tag{7}$$

under the restriction that the same λ^n are used for all the Q sub-problems in Equation (7).

Proof:

We have,

$$\begin{aligned} &\min_{x \in \Delta} \left[C(x) + \sum_{n=1}^{N} \lambda^n Cr^n(x) \right] \\ &= \min_{x \in \Delta} \left[\sum_{q=1}^{Q} C_q(x_q) + \sum_{n=1}^{N} \lambda^n \sum_{q=1}^{Q} Cr_q^n(x_q) \right]. \end{aligned} \tag{8}$$

According to Fubini theorem, one may write:

$$\begin{aligned} &\min_{x \in \Delta} \left[C(x) + \sum_{n=1}^{N} \lambda^n Cr^n(x) \right] \\ &= \min_{x \in \Delta} \left[\sum_{q=1}^{Q} C_q(x_q) + \sum_{q=1}^{Q} \sum_{n=1}^{N} \lambda^n Cr_q^n(x_q) \right] \end{aligned} \tag{9}$$

$$= \min_{x \in \Delta} \sum_{q=1}^{Q} \left[C_q(x_q) + \sum_{n=1}^{N} \lambda^n Cr_q^n(x_q) \right]. \tag{10}$$

Since the term $[C_q(x_q) + \sum_{n=1}^{N} \lambda^n Cr_q^n(x_q)]$ is always positive, and since choosing x_q is, besides the economical interdependence due to the limited budgets, independent from choosing x_l for $q \neq l$, one can write:

$$\begin{aligned} &\min_{x \in \Delta} \sum_{q=1}^{Q} \left[C_q(x_q) + \sum_{n=1}^{N} \lambda^n Cr_q^n(x_q) \right] \\ &= \sum_{q=1}^{Q} \min_{x_q \in \Delta_q} \left[C_q(x_q) + \sum_{n=1}^{N} \lambda^n Cr_q^n(x_q) \right] \end{aligned} \tag{11}$$

where Δ_q is the set of possible strategies for structure q.

The economical interdependence due to the imposed budgets at the beginning of each time period is translated into the mathematical formulation by means of the constraint that the same values of λ^n must be used for all the Q sub-problems.

Since it was assumed that the evolution of the states of each structure has the Markovian property, i.e. $\min_{x_q \in \Delta_q} C_q(x_q)$can be solved by dynamic programming; then problem (7) can be also solved by dynamic programming. For that purpose, one must simply add the term $\lambda^n Cr_q^n(x_q)$ to the direct costs in the dynamic programming recursive relation if the calculated expected long-term minimal cost is not discounted, or one must add the term $(1+\alpha)^n \lambda^n Cr_q^n(x_q)$ if the calculated expected long-term minimal cost is discounted at a discount rate α. Hence, for a given set of values of λ^n, MDP or GPOMDP algorithms should be executed for each structure. Then, one compares the $Cr^n(*x)$ with the bl^n and modifies the λ^n accordingly until a set of λ^n values is found so that $Cr^n(*x)$ becomes equal to bl^n for all the time periods. The existence of a set of values for λ^n that respect the imposed constraints and the procedures used for the determination of the values of λ^n and $Cr^n(*x)$ are presented in the next section.

Existence and computation of Lagrange multipliers λ^n

As mentioned in the preceding section, problem (4) can be answered by solving repeatedly problem (7) for various values of λ^n until one obtains $Cr^n(^*x) = bl^n$ for $n = 1, \dots, N$. The practical computation of $Cr^n(^*x)$ will be discussed in a later section. Nevertheless, a legitimate question arises about the existence of an optimality gap i.e. an optimal solution satisfying exactly the constraints does exist but the proposed methodology may not be able to find it. An optimality gap in Lagrange relaxation optimisation is due to the non-convexity of the problem (Everett 1963, Fisher 1981). For our particular problem, two basic causes can lead to such a possibility: First, when the use of one resource requires the use of other resources, or equivalently in cases where some constraints may involve various combinations of other constraints (Everett 1963). As for the particular case of our problem, such a limitation can be avoided if one assumes that the costs of the possible inspection and maintenance actions that can be undertaken during each time period are allocated to the budget of that time period only. Second, if the set of strategies x over which the objective function is defined, is discrete (which is the case in our study); then in general, there will be an optimality gap since integer programs are non-convex. In that case, the proposed methodology ushers to solutions which form an upper and lower bounds to the real optimal solution. However, these solutions will be optimal for their associated levels of resource consumption (Everett 1963). In particular, if the imposed budgets are equal to the expected expenditures required by these solutions, then these solutions will be the optimal ones. Such a feature is a reassuring property since it guarantees us that if an exact solution is found, then it is the optimal one. In a later section, the practical considerations associated with this issue will be discussed in more detail.

Concerning the computation of the λ^n values, it should be emphasised here that the term $\sum_{n=1}^{N} \lambda^n Cr_q^n(x)$ in problem (7) can be thought as a sum of penalty functions (Bazaraa *et al.* 2006). Hence, if all but one λ^n is held constant, the resource that changes is a monotonically decreasing function of its associated multiplier. Based on this remark, a simple procedure is proposed for finding the appropriate set of $\lambda^{n'}$s for which $Cr^n(^*x) \approx bl^n$ where $n = 1, \dots, N$. This procedure is presented in Appendix.

GPOMDP decision tree for the multi-structure case

The aim of this section is to illustrate how to modify a discounted GPOMDP decision tree to include the term $(1+\alpha)^n \lambda^n Cr_q^n(x)$ for the particular case of a GPOMDP having a decision tree of the type illustrated in Figure 1, i.e. the sequence of actions to be optimised during each time period consists of one inspection followed by one maintenance action. Similar modifications are straightforward for any other GPOMDP having a different decision tree structure.

In this article, each structure of the group may have a specific decision tree which is compatible with the maintenance actions and inspection techniques that are suitable to that particular structure.

For the particular case of a decision tree having a sequence of two decisions (one inspection followed by one action) only, the total expected cost $c(v^n)_{|i,r,a^n,\theta^n}$ depends on the inspection technique i, the vector of the inspection results r, the action a^n and, the state of the structure θ^n. Notice that θ^n concentrates the probability mass in one component of the vector v^n. The conditional θ^n will be eliminated in subsequent calculations by integration (expected value). Thus:

$$
\begin{aligned}
c(v^n)_{|i,r,a^n,\theta^n} = \\
&= [ci(i) + ca(a^n)] \times [(1+\alpha)^n \lambda^n + 1] \\
&\quad + \frac{1}{1+\alpha} \times {}^*c\left(v^{n+1} | i, r, a^n, v^n\right) + \mathrm{E}_{^a\theta^n} | \theta^n, a^n [cs(^a\theta^n)] \\
&= [ci(i) + ca(a^n)] \times [(1+\alpha)^n \lambda^n + 1] \\
&\quad + \frac{1}{1+\alpha} \times {}^*c\left(v^{n+1} | i, r, a^n, v^n\right) \\
&\quad + \sum_k cs\left(^a\theta_k^n\right) \times a_{jk}^{an}. \\
&= [ci(i) + ca(a^n)] \times [(1+\alpha)^n \lambda^n + 1] \\
&\quad + \frac{1}{1+\alpha} \times {}^*c('v^n \times A_{an} \times M) + \sum_k cs\left(^a\theta_k^n\right) \times a_{jk}^{an}
\end{aligned} \tag{12}
$$

In this equation, a penalising term $[ci(i) + ca(a^n) \times (1+\alpha)^n \lambda^n]$ on the costs of maintenance actions and inspections is introduced. The term $ci(i) + ca(a^n)$ represents the costs of IM&R decisions. The term $\frac{1}{1+\alpha} \times {}^*c\left(v^{n+1} | i, r, a^n, v^n\right)$ represents the discounted cost associated with the belief state at the beginning of stage $n+1$ given that the belief state at the beginning of period n was v^n, that we applied inspection i, got the result r and applied action a^n. By applying inspection and getting its result, v^n can be updated by using the Bayes formula to get $'v^n$. v^{n+1} will be $v^{n+1} = {}'v^n \times A_{an} \times M$. The term $\mathrm{E}_{^a\theta^n} | \theta^n, a^n [cs(^a\theta^n)]$ represents the expected cost to be paid by the users due to deck condition. Knowing the state $\theta^n = \theta_j^n$ and the action a^n,$^a\theta^n$ will have the probability distribution $\Pr(^a\theta_k^n) = a_{jk}^{an}$.

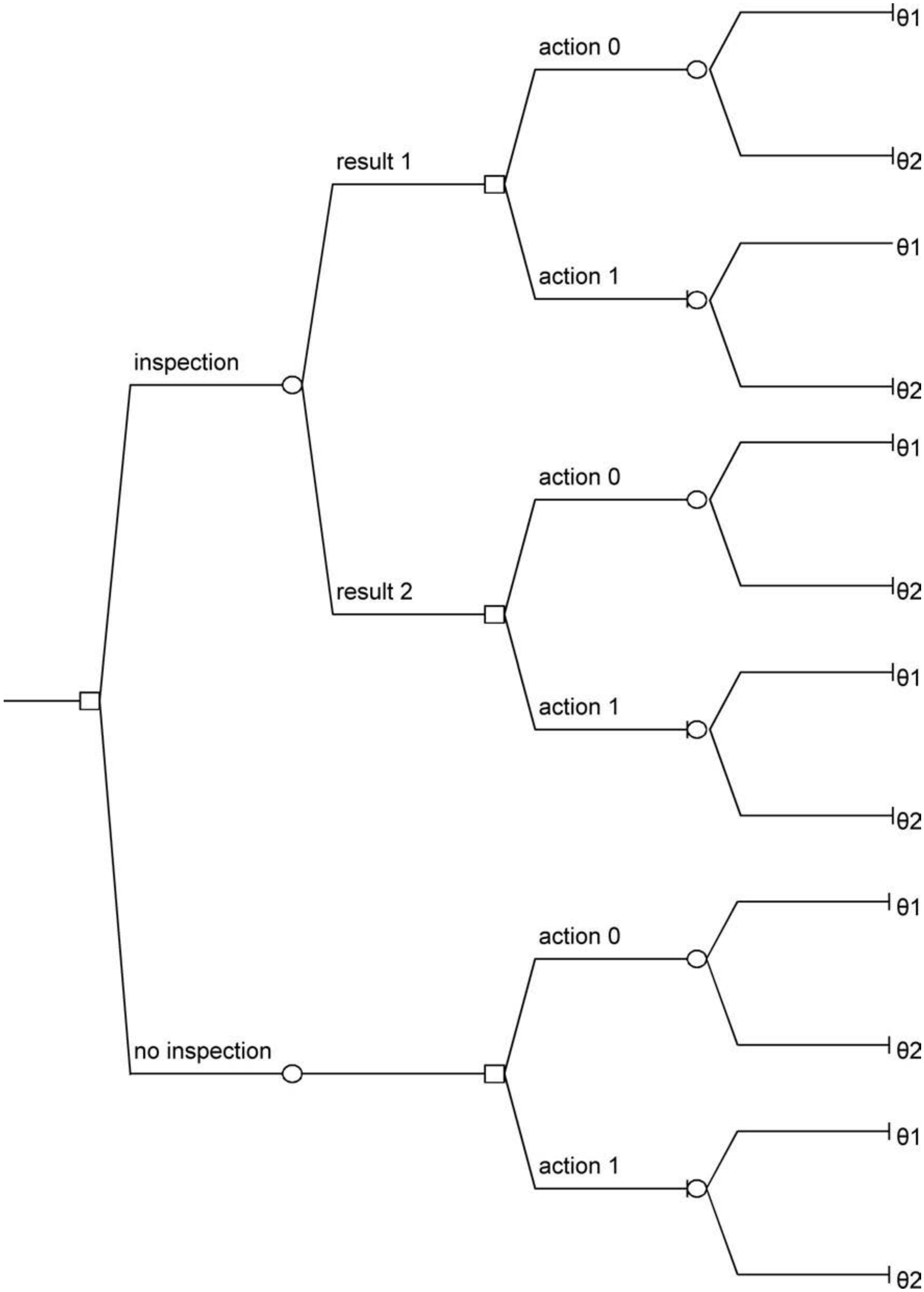

Figure 1. Example of a decision tree for inspections followed by maintenance actions.

Hence, $E_{^a\theta^n}|\theta^n, a^n[cs(^a\theta^n)] = \sum_k cs(^a\theta^n_k) \times a^{an}_{jk}$. The decision analysis calculations will be:

$$\begin{aligned} c(v^n)_{|i,r,a^n} &= E_{\theta^n} c(v^n)^n_{|i,r,a^n,\theta_n} \\ &= E_{\theta^n}\Big\{ [ci(i) + ca(a^n)] \times [(1+\alpha)^n \lambda^n + 1] \\ &\quad + \frac{1}{1+\alpha} \times {}^* c('v^n \times A_{an} \times M) \\ &\quad + \sum_k cs(^a\theta^n_k) \times a^{an}_{jk} \Big\} \\ &= [ci(i) + ca(a^n)] \times [(1+\alpha)^n \lambda^n + 1] \\ &\quad + \frac{1}{1+\alpha} \times {}^* c('v^n \times A_{an} \times M) \\ &\quad + \sum_j \Big[\sum_k cs(^a\theta^n_k) \times a^{an}_{jk} \Big] \times \Pr[\theta^n_j | r, i] \end{aligned} \quad (13)$$

The optimal expected cost corresponding to belief state v^n is given by:

$$^*c(v^n)_{|i,r} = \min_{a^n \in A} c(v^n)_{|i,r,a^n}. \quad (14)$$

Given the vector of results r of the inspection type i, the optimal action will be:

$$^*a^n | r, i = \arg\min_{a^n \in A} c(v^n)_{|i,r,a^n} \quad (15)$$

$$^*c(v^n)_{|i} = E_r\left[^*c(v^n)_{|i,r}\right] = \sum_l {}^*c(v^n)_{|i,r} \times \Pr(r_l) \quad (16)$$

It should be noted that $^*c(v^n)_{|i,r}$ depends on the vector of inspection results r via the posterior belief state $'v^n$ which is calculated based on r.

$$^*c(v^n) = \min_{i \in I} {}^*c(v^n)_{|i}. \quad (17)$$

The optimal inspection type is:

$$^*i = \arg\min_{i \in I} {}^*c(v^n)_{|i}. \quad (18)$$

It should be noted that one can specify an inspection type i_0 having a zero cost consisting in doing nothing. This will stand in our model for the option 'no inspection to be done'.

Computation of $Cr^n(x)$

For every belief state vector v^n at the beginning of each time period, the decision tree of the GPOMDP will prescribe a sequence of optimal decisions. The selection of some of the actions in the sequence will depend on the result of the inspection decisions made earlier in the sequence. In the planning phase, prior probability distributions can be assigned to the different possible results of the planned inspections. Hence, for each belief state vector v^n during time period n, the decision tree will provide us with the expected direct cost $Cr^n_q(x)$ to be paid by the manager for IM&R decisions during that period. Besides, based on the mentioned prior probability distributions of the inspection results, and since specific IM&R actions are uniquely determined by v^n and by the inspection results, one is able to calculate for each v^n a probability distribution of the belief states v^{n+1}. Therefore, if for each belief state v^{n+1} one has already calculated the corresponding $Cr^k_q(x)$ for $k = n+1, \ldots, N$, one will be able to calculate for each v^n, based on the probability distribution of the states v^{n+1}, an expected value of the $Cr^k_q(x)$ for $k = n, \ldots, N$. Doing so recursively, one will be able to calculate for v^1 the expected values $Cr^k_q(x)$ for $k = 1, \ldots, N$. Consequently, since each of the structures in the inventory has a unique known belief state at the beginning of the planning time, one will be able to determine for each structure q the expected values $Cr^k_q(x)$ for $k = 1, \ldots, N$. Finally notice that the resources expended by the whole inventory during time period n will be:

$$Cr^n(x) = \sum_q Cr^n_q(x). \quad (19)$$

Solution methods

For each structure in the inventory, it is assumed that a belief state is known by the manager. This belief state results from previous inspections and/or it is the outcome of a collection of deterioration models applied to some of the structures. In the following, three alternatives solution methods based on the iteration procedure discussed previously (cf. Appendix) are presented.

Solution method 1

Bearing in mind previous discussion, the problem of optimising IM&R decisions for an inventory of structures over a finite time horizon and assuming that the evolution of the states of each structure is Markovian can be solved using the following steps:

(0) Apply procedure lambda for each structure. It should be mentioned that solving problem 7 using procedure lambda is equivalent to solving a modified GPOMDP (or any other modified dynamic programming algorithm);

(1) Apply the sequences of decisions that are prescribed by the modified GPOMDP to each structure of the group.

Solution method 2

Since some of the sequences of actions that are prescribed by the modified GPOMDP to each structure of the group begin in general by an inspection other than i_0 (i.e. no inspection); then, one can use the fact that the whole inventory has a new information state after the inspection of some of its structures. In order to be able to take advantage of this updated information state of the inventory, one is compelled to perform all the requested inspections and then to solve the whole problem again (with the remaining budget for the current time period) based on the newly available information before applying an M&R action. As such, solution method 2 will be identical to solution method 1, apart from step 1 which must be replaced by the following two steps:

(1) If no inspection is prescribed for any of the structures, goto step 2 given below, else execute the inspections prescribed by the results of the modified GPOMDP to each structure of the group (when applicable). The condition states of the inspected structures will have new posterior probability distributions, i.e. new belief states. The remaining budget for the current time period will be $bl^1 = bl^1 - \sum_{k=1}^{M} ci(i_k)$; goto step 0.
(2) Execute the M&R actions prescribed for the current time period.

It should be noted that the above listed procedure does not mean that an inspection other than i_0 (no inspection) has to be necessarily done for all the structures.

Solution method 3

The optimality of solution method 2 can be enhanced by splitting the set of prescribed inspections in step 1 into smaller subsets. Thus, instead of executing all the prescribed inspections, one executes the inspections contained in only one of the subsets. On the basis of the new information state of the whole inventory, one solves the whole problem again (with the remaining budget for the current time period) before applying any M&R action. The rationale of doing so lies in the fact that based on the information resulting from a small subset of the original set of prescribed inspections, some of the previously prescribed inspections may not be prescribed by the new solution and hence, their costs will be saved. As such, solution method 3 will have the same steps as solution method 2 apart from step 1 which must be replaced by the following:

(1) If no inspections are prescribed goto step 2, else: execute some selected inspections prescribed by the results of the modified GPOMDP of each structure. The condition states of the inspected structures will have new posterior probability distributions, i.e. new belief states. The remaining budget for the current time period will be $bl^1 = bl^1 - \sum_{k=1}^{M} ci(i_k)$; goto step 0.

However two questions remain to be answered:

(1) How small should the selected subset of prescribed inspections be?
(2) Which subset should be chosen among all possible subsets?

Clearly, the smaller the selected subset is, the lower the expected total cost is. This means that to be optimal, the inventory manager should select only one inspection at a time. However, time scheduling considerations and logistical constraints (among others) will not allow the manager to opt for such an extreme case. An approximate solution to the second question will be to select a subset of prescribed inspections having the maximum value of information over inspection cost ratio.

Finally, it should be noted that solution method 1 can be used with any type of MDP or POMDP; however, solution methods 2 and 3 can only be used with POMDPs allowing one to obtain optimal inspections. Models of this type are Madanat and Ben-Akiva model (1994), Corotis *et al.* (2005) model and GPOMDP by Faddoul *et al.* (2009).

Practical considerations and computational complexity

Our optimisation problem can be considered as an integer programming problem due to the fact that the maintenance actions and inspection techniques are selected from finite countable sets of alternatives. Therefore, our problem is not convex. Hence, as mentioned earlier, there will be an optimality gap where no possible set of λ^n, which correspond to the real optimal solution, exists. However, in a cell problem, as the number of cells increases (number of structures for our particular case), the result of over-all optimisation is a total problem in which concavities are vastly reduced in significance (Everett 1963). This can be intuitively explained by the fact that as the number of structures increases the cost of available inspections and/or maintenance actions for each structure will be relatively small compared to the available budg*et al*located to the whole inventory; hence the problem, which is discrete, begins to emulate the behaviour of a continuous problem. Hence, the larger is the inventory, the smaller the relative discrepancy will be. By the same logic, if the number of available inspections and maintenance actions for each structure is large the effects of non-convexity will be diminished. Moreover, when we are dealing with POMDPs, where the state of the structure is a probability distribution and the actions are probabilistic and chosen on the basis of the belief state which is possibly updated by the uncertain results of imperfect inspections, then the prescribed expenditures during each time period are expected expenditures. In other terms, they are a kind of weighted average of the costs of the different available inspections and maintenance actions. In that case, if we disregard the first decision at the beginning of the first time period which evidently affects the following decisions in the following time periods, the problem is not really integer since the weighted average can be adjusted on an almost continuous scale especially for the future time periods. Hence, in the case of POMDPs with decisions having

Table 1. Costs due to deck condition.

State ${}^{a}\theta^{n}$	Condition of the deck	The costs (in arbitrary units) incurred during each stage due to the deck condition, $cs({}^{a}\theta^{n})$
1	The deck is in very good condition	200 units
2	The deck is in good condition	600 units
3	The deck condition is in fairly good condition	1250 units
4	The deck is in poor condition	2000 units
5	The deck is in very poor condition	3500 units

Table 2. Elements of the deterioration matrices for each bridge.

	b_1 *to* b_6	b_7	b_8	b_9	b_{10}	b_{11} *to* b_{16}
m_{11}	0.5	0.6	0.5	0.5	0.5	0.5
m_{12}	0.25	0.2	0.25	0.25	0.25	0.25
m_{13}	0.2	0.2	0.2	0.2	0.2	0.2
m_{14}	0.05	0	0.05	0.05	0.05	0.05
m_{15}	0	0	0	0	0	0
m_{21}	0	0	0	0	0	0
m_{22}	0.5	0.5	0.6	0.5	0.5	0.5
m_{23}	0.25	0.25	0.2	0.25	0.25	0.25
m_{24}	0.2	0.2	0.2	0.2	0.2	0.2
m_{25}	0.05	0.05	0	0.05	0.05	0.05
m_{31}	0	0	0	0	0	0
m_{32}	0	0	0	0	0	0
m_{33}	0.5	0.5	0.5	0.6	0.5	0.5
m_{34}	0.3	0.3	0.3	0.2	0.3	0.3
m_{35}	0.2	0.2	0.2	0.2	0.2	0.2
m_{41}	0	0	0	0	0	0
m_{42}	0	0	0	0	0	0
m_{43}	0	0	0	0	0	0
m_{44}	0.7	0.7	0.7	0.7	0.8	0.7
m_{45}	0.3	0.3	0.3	0.3	0.2	0.3
m_{51}	0	0	0	0	0	0
m_{52}	0	0	0	0	0	0
m_{53}	0	0	0	0	0	0
m_{54}	0	0	0	0	0	0
m_{55}	1	1	1	1	1	1

Table 3. Inspection costs for each bridge.

	b_1 *to* b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}
i_0	0	0	0	0	0	0
i_1	20	15	10	20	10	10
i_2	40	20	15	40	40	15
i_3	40	40	15	40	40	15

Table 4. Conditional probability distribution of the results of inspection i_1.

$\mathrm{P}(r_{\mathrm{m}}\vert\ \theta^{n})$		r_1	r_2	r_3	r_4	r_5
θ^{n}	1	0.4	0.3	0.15	0.1	0.05
	2	0.25	0.4	0.2	0.1	0.05
	3	0.1	0.2	0.4	0.2	0.1
	4	0.05	0.1	0.2	0.4	0.25
	5	0.05	0.1	0.15	0.3	0.4

Table 5. Conditional probability distribution of the results of inspection i_2.

$P(r_{\mathrm{m}}\vert\ \theta^{n})$		r_1	r_2	r_3	r_4	r_5
θ^{n}	1	0.8	0.15	0.05	0	0
	2	0.1	0.8	0.1	0	0
	3	0.05	0.1	0.7	0.1	0.05
	4	0.05	0.1	0.15	0.5	0.2
	5	0.05	0.1	0.15	0.2	0.5

Table 6. Conditional probability distribution of the results of inspection i_3.

$P(r_{\mathrm{m}}\vert\ \theta^{n})$		r_1	r_2	r_3	r_4	r_5
θ^{n}	1	0.5	0.2	0.15	0.1	0.05
	2	0.2	0.5	0.15	0.1	0.05
	3	0.05	0.1	0.7	0.1	0.05
	4	0	0	0.1	0.8	0.1
	5	0	0	0.05	0.15	0.8

Table 7. Cost of the maintenance actions for each bridge.

	Action type	Costs $ca(a_i)$ (units)						
		Bridges 1–10	Bridge 11	Bridge 12	Bridge 13	Bridge 14	Bridge 15	Bridge 16
a_0	Nothing to do	0	0	0	0	0	0	0
a_1	Preventive maintenance	800	600	700	500	500	650	700
a_2	Corrective maintenance	800	750	800	750	500	400	500
a_3	Replacement	3000	2500	2000	3000	2000	2700	1000

Table 8. Markov transition matrices for actions a_0, a_1, a_2 and a_3.

A0 =

	${}^a\theta_1^n$	${}^a\theta_2^n$	${}^a\theta_3^n$	${}^a\theta_4^n$	${}^a\theta_5^n$
θ_1^n	1	0	0	0	0
θ_2^n	0	1	0	0	0
θ_3^n	0	0	1	0	0
θ_4^n	0	0	0	1	0
θ_5^n	0	0	0	0	1

Transition matrix for action a_0

A1 =

	${}^a\theta_1^n$	${}^a\theta_2^n$	${}^a\theta_3^n$	${}^a\theta_4^n$	${}^a\theta_5^n$
θ_1^n	1	0	0	0	0
θ_2^n	0.7	0.3	0	0	0
θ_3^n	0.4	0.4	0.2	0	0
θ_4^n	0	0.2	0.3	0.4	0.1
θ_5^n	0	0	0.3	0.3	0.4

Transition matrix for action a_1

A2 =

	${}^a\theta_1^n$	${}^a\theta_2^n$	${}^a\theta_3^n$	${}^a\theta_4^n$	${}^a\theta_5^n$
θ_1^n	0.9	0.1	0	0	0
θ_2^n	0.15	0.7	0.15	0	0
θ_3^n	0.1	0.3	0.5	0.1	0
θ_4^n	0.4	0.3	0.2	0.1	0
θ_5^n	0.2	0.4	0.2	0.1	0.1

Transition matrix for action a_2

A3 =

	${}^a\theta_1^n$	${}^a\theta_2^n$	${}^a\theta_3^n$	${}^a\theta_4^n$	${}^a\theta_5^n$
θ_1^n	1	0	0	0	0
θ_2^n	1	0	0	0	0
θ_3^n	1	0	0	0	0
θ_4^n	1	0	0	0	0
θ_5^n	1	0	0	0	0

Transition matrix for action a_3

uncertain results, the methodology will always find a solution which is very near to the optimal one. Finally, for problems having a significant optimality gap, a number of handling methods are available such as those suggested by Everett in his paper. We will not detail these methods in this article.

In our implementation, the terminating condition $Cr^n(^*x) \approx bl^n$ of procedure lambda is considered as satisfied when one of the two following criterions occurs: The first criterion is $\left|\frac{Cr^n(^*x)-bl^n}{bl^n}\right| < \mu$ where μ is a chosen arbitrary threshold under which the terms $Cr^n(^*x)$ and bl^n are considered to be approximately equal. The second one is considered as satisfied when $Cr^n(^*x)$ alternates between two unchanging values of x_1 and x_2 such that $x_1 < bl^n < x_2$ for an increasingly finer adjustment of the values of λ^n. In that case, the gap between x_1 and x_2 is due to the integer nature of the problem. If the budgetary constraints are hard in the sense that they have to be strictly satisfied (as for example for budgets from private societies or others); one should adopt the λ^n value that corresponds to the resource consumption value x_1. Empirical, experimental and numerical computations performed using the present model have shown (not presented in this article) that in an inventory containing more than a dozen of structures, the relative discrepancy (due to the integer nature of the problem) between budget limits and the obtained $Cr^n(x^*)$ is, in the vast majority of time, less than 2%.

Finally, notice that the complexity of the proposed algorithm is proportional to the number M of facilities in the system, i.e. $O(M)$. As for the impact of the number of budget constraints on the computation time, one should note that the search for the appropriate set of λ^n is not a brute force search since at the end of each iteration, the value of each λ^n will be changed according to the sign and magnitude in procedure of the relative discrepancy of its associated level of resource consumption. As such, the number of iterations needed for finding a set of λ^n satisfying the terminating condition $Cr^n(^*x) \approx bl^n$ is usually very small. For example, if μ is equal to 2% the number of iterations needed for a seven-time period problem, viz.

having seven budgetary constraints, is in the order of 8–10 iterations; whereas the number needed for a 25 budgetary constraints problem is in the order of 10–12 iterations. Moreover, the proposed methodology is very suitable to parallel processing. If for example a processing unit is assigned to each of the parallel MDPs, i.e. to each structure, then the amount of information to be communicated between these units after each iteration is $2 \times N$ real numbers.

Numerical application

Consider a set of 16 highway concrete bridges. It is assumed that the performance of the concrete deck of each bridge is described by five states as indicated in Table 1. The cost $cs(^a\theta^n)$ of a bridge caused by the deck conditions (cf. Table 1) is generally the sum of the costs incurred by the users of this bridge due to malfunctioning of the deck, in addition to the risk which is expressed in monetary units and calculated as the product of the failure probability of the bridge by the costs generated by such a failure. The stochastic deterioration of each bridge is modelled by a Markov chain. The Markovian deterioration transition matrices for the different bridges (i.e. b_1 to b_{16}) are detailed in Table 2. It is assumed that four imperfect inspection techniques are available, where i_0 means that no inspection is performed (i.e. one entirely relies on the prediction of the deterioration model) and its cost is 0 unit. The costs of the inspection techniques for each bridge are detailed in Table 3. The possible results of an inspection are denoted by r_i $(i = 1, 2, 3, 4, 5)$ where r_i means that the deck is in state θ_i. The uncertainties associated with the results of the inspection techniques i_1, i_2 and i_3 are expressed by the probability distributions shown in Tables 4–6. As it

Table 9. Initial belief states.

P (θ^1)						
θ^1	Bridges 1, 7, 11	Bridges 2 8, 12	Bridges 3, 9, 13	Bridges 4, 10, 14	Bridges 5, 15	Bridges 6, 16
1	0.2	1	0	0	0	0
2	0.3	0	1	0	0	0
3	0.3	0	0	1	0	0
4	0.2	0	0	0	1	0
5	0	0	0	0	0	1

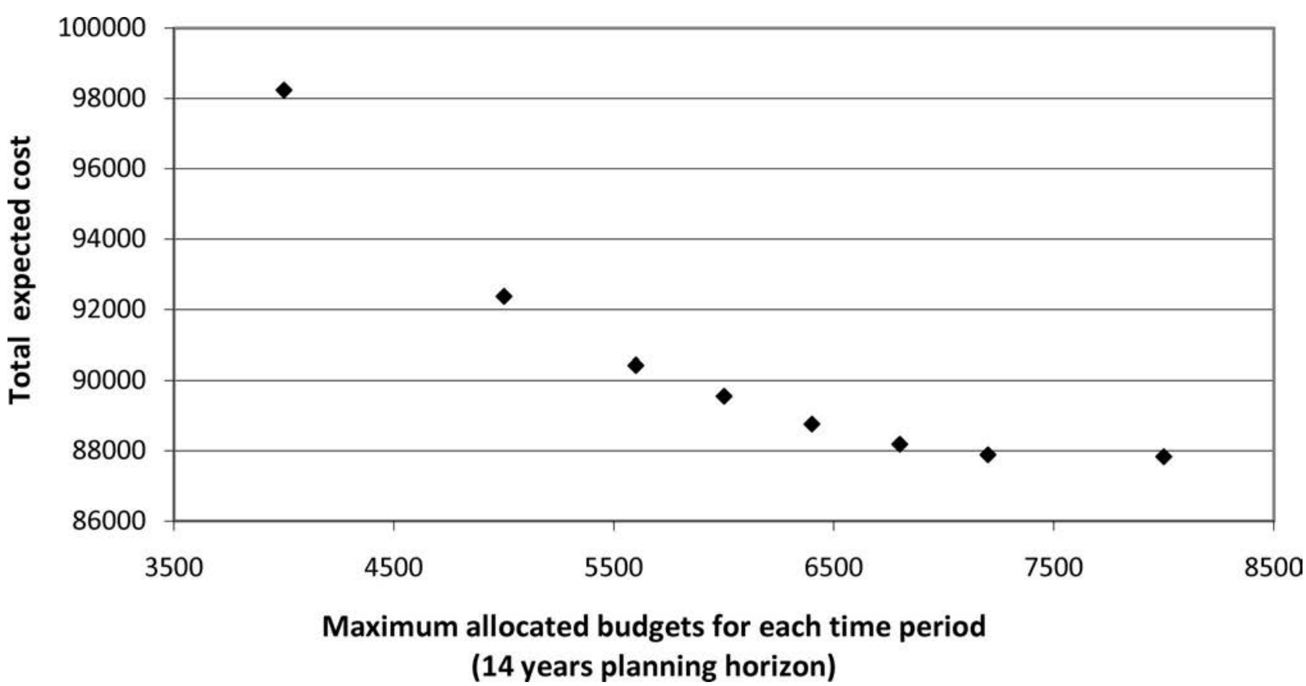

Figure 2. Impact of the maximum allocated budgets on the total expected cost (14 years time horizon).

Table 10. Expected costs for budgets varying from 8000 to 6400 units.

bl^n:Available budget	8000		7200		6800		6400	
:Available budget	λ^n	$Cr^n(x^*)$	λ^n	$Cr^n(x^*)$	λ^n	$Cr^n(x^*)$	λ^n	$Cr^n(x^*)$
Period 1	0	7184.65	2.77E-02	7184.65	0.2161	6866.75	0.4993	6425.35
Period 2	0	7650.54	6.46E-02	7119.88	0.1451	6828.52	0.2858	6461.31
Period 3	0	7514.14	7.58E-02	7248.21	0.2099	6798.51	0.2761	6418.77
Period 4	0	7720.08	8.15E-02	7254.23	0.1927	6788.47	0.2442	6402.28
Period 5	0	7351.04	5.70E-02	7208.4	0.144	6820.42	0.1881	6393.69
Period 6	0	7058.48	6.55E-03	7229.42	6.41E-02	6749.1	9.22E-02	6394.35
Period 7	0	3651.65	0.00E + 00	3709.51	0	4009.45	0	4256.6
Sum of the discounted expected direct costs	34152.65		33234.6		31692.46		30101.9	
$C(x)$	87824.77		87882.28		88181.32		88756	

Note: Available budgets are assumed to be similar for the different time periods.

Table 11. Expected costs for budgets varying from 6000 to 4000 units.

bl^n	6000		5600		5000		4000	
:Available budget	λ^n	$Cr^n(x^*)$	λ^n	$Cr^n(x^*)$	λ^n	$Cr^n(x^*)$	λ^n	$Cr^n(x^*)$
Period 1	0.5412	5925.35	0.6989	5822.75	0.7968	4834.75	1.2148	3976.31
Period 2	0.3535	6040.35	0.4396	5496.15	0.592	5241.62	1.1849	4096.39
Period 3	0.343	6051.04	0.4332	5608.74	0.6014	4996.16	1.3069	4069.25
Period 4	0.3026	6068.21	0.3969	5592.84	0.5577	5052.55	1.3169	4043.45
Period 5	0.2359	6021.93	0.3139	5604.87	0.4685	5070.77	1.1449	4017.4
Period 6	1.26E-01	6027.38	1.84E-01	5580.11	3.15E-01	5047.99	7.98E-01	4001.69
Period 7	0	4501.82	0	4831.47	3.66E-02	5022.77	0.2792	3998.56
Sum of the discounted expected direct costs	28484.47		26943.97		24474.86		19598.67	
$C(x)$	89547.47		90423.26		92367.69		98232.82	

Note: Available budgets are assumed to be similar for the different time periods.

can be noted from the numerical values in Tables 5 and 6, inspection i_2 is specialised in detecting the states θ_1 and θ_2 while inspection i_3 is specialised in detecting the states θ_4 and θ_5 of the structure. It is assumed that only three maintenance actions can be performed (Table 7). The uncertainties associated with the consequences of maintenance actions are expressed by the matrices shown in Table 8. The element a_{ij} in each of these matrices corresponds to the probability that the deck, which is initially in state $\theta^n = i$ (before the maintenance action), will be after the application of a maintenance action in state ${}^a\theta^n =$ j (the superscript a means that the state into consideration occurs immediately after the maintenance action before any deterioration can take place). The initial belief states for the different bridges are detailed in Table 9. The length of each stage was taken to be 2 years long and the discount rate was taken $\alpha = 0.049$. The planning horizon is set to 14 years.

The results were obtained using specialised GUI software for GPOMDPs that we have developed. Although our software is designed to deal with different allocated budgets bl^n for the different time periods; for simplicity, the budgets are assumed to be equal in the present example. The threshold μ was set to 2%. The optimal expected total costs for the whole inventory $C(x)$ corresponding to different values of the allocated budgets ranging from 4000 to 8000 units (for each time period) are presented in Figure 2. In Tables 10 and 11 are also presented (i) the direct costs $Cr^n(*x)$ associated with each budget limit as well as the corresponding values of λ^n, for the different time periods and (ii) the corresponding discounted sums of the direct costs, i.e. $\sum_{n=1}^{N} 1/(1+\alpha)^n Cr^n({}^*x)$. It can be noted that the total cost for the whole inventory $C(x)$ increases when one limits further the available budget for each time period. However, it can be noted that this increase in the total cost is very slow at the beginning;

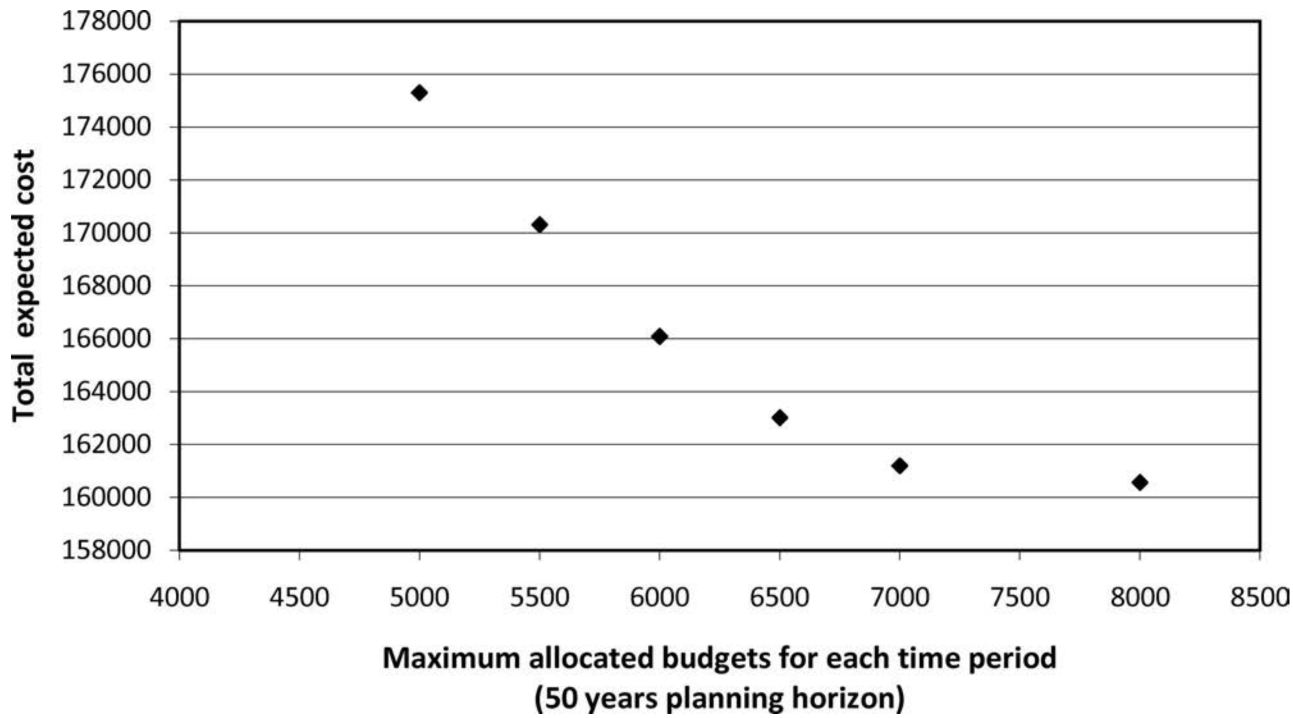

Figure 3. Impact of the maximum allocated budgets on the total expected cost (50 years time horizon).

then, it increases exponentially as the imposed limiting budget decreases further. Moreover, one can note that when one decreases the limiting budget from 8000 to 6000 units for example, the total discounted expected direct cost to be paid by the manager is decreased from 34152.7 to 28484.5 that is by 5668.2 units, while the discounted total cost increased by only 89547.47 – 87824.77 = 1722.7 units. The discrepancy between the two numbers, namely 5668.2 – 1722.7 = 3945.5 units, is due to the fact that this portion of the total cost for all the structures was transferred from the manager to the users (remember here that the total cost includes the term $cs({}^a\theta^n)$ which is the cost incurred by the structure due to a degraded performance for being in state ${}^a\theta^n$ during stage n, after the application of the action). This means that, as the manager limits the available funds to maintain the structures, a larger part of the total cost is transferred to the users of the structures. Concerning the flat portion of the curve (i.e. as long as the allocated budget is larger than 6500 units), such a transfer is not a real inconvenience since the total expected cost does not significantly increase. One can even argue that such ability to transfer costs can be used suitably by managers or governments who have a shortage of funds. Besides, one can note that for budgets of 8000, 7200, 6800, 6400, 6000 and 5600 units, $\lambda^7 = 0$ since the required sum is lower than the available budget. However, it can be noted that as

Table 12. Expected costs for budgets of 6000 units.

	λ^n	$Cr^n(x^*)$
Period 1	0.7122626959	5995.47
Period 2	0.5063214576	5987.72
Period 3	0.5233051267	6005.43
Period 4	0.5118336299	6003.72
Period 5	0.4831212022	6003.08
Period 6	0.4516625846	6024.37
Period 7	0.4184826364	6009.28
Period 8	0.3859702313	5999.46
Period 9	0.3541740982	6002.06
Period 10	0.3235476236	6008.50
Period 11	0.2957264436	5977.22
Period 12	0.2697490319	5944.28
Period 13	0.2440967348	6052.88
Period 14	0.2236951594	5936.40
Period 15	0.202850768	5977.74
Period 16	0.1847729555	5928.02
Period 17	0.1651636624	6047.10
Period 18	0.1497547219	5935.11
Period 19	0.1333040538	6008.87
Period 20	0.11748382	6019.40
Period 21	0.1012087941	6034.57
Period 22	8.48905959e-002	5950.23
Period 23	6.23027078e-002	6032.43
Period 24	3.249999928e-002	5945.39
Period 25	0	4355.94
Sum of the discounted expected direct costs		54111.77
$C(x)$		166086.23

Note: Available budgets are assumed to be similar for the different time periods.

the allocated budget decreases from 8000 to 5600 units, the required sum for IM&R in period 7 increases from 3651.65 to 4831.47 units. This is due to the poor condition state of the inventory at period 7 which was caused by insufficient budgets in previous time periods. The calculations were also made for a planning time horizon of 50 years (Figure 3 and Table 12).

Conclusion

A methodology for extending MDPs or POMDPs (using dynamic programming methods) from single structure to multiple-structure levels is presented in this article for the optimisation of IM&R of civil engineering infrastructures. The whole optimisation problem was transformed into a sum of smaller problems, one for each structure. The methodology was illustrated by using a GPOMDP. A Lagrange relaxation technique was employed to overcome the complexity resulting from the economic interdependence between IM&R decisions for all structures. The existence of an appropriate set of Lagrange multipliers was discussed and a procedure for finding these multipliers was presented. The proposed methodology differs from those based on linear programming usually used to solve parallel MDPs in the sense that it allows the different structures of each process to be dissimilar. Also, three alternative solution methods were used for optimising IM&R decisions for the whole inventory. They differ from each other according to the way they use the information produced by the prescribed inspections. A numerical application illustrating the model is given where sensitivity analysis for varying maximum allocated budgets for each time period is carried out.

Notation

θ	State of the structure;
θ^n	State of the structure at the beginning of time period n;
${}^a\theta^n$	State of the structure during time period n after a maintenance action a has been applied;
v^n	Belief state vector of the structure at the beginning of time period n;
${}^av^n$	Belief state of the structure during time period n after a maintenance action a has been applied;
${}'v^n$	Bayesian posterior belief state vector;
a^n	Action to be applied at the beginning of time period n;
i^n	Inspection to be applied at the beginning of time period n;
$*a^n$	Optimal action to be applied at the beginning of time period n;
$*i^n$	Optimal inspection to be applied at the beginning of time period n;
M	Markov chain transition matrix;
A_{an}	Transition matrix from θ^n to ${}^a\theta^n$ related to action a^n;
a_{jk}^{an}	Element of row j and column k of matrix A_{an};
$ci(\mathrm{i})$	Cost associated with the application of the inspection method i;
$ca(a)$	Cost of action a;
$cs({}^a\theta^n)$	Cost incurred by the structure for being in state ${}^a\theta^n$ during time period n after application of action a;
$cs({}^av^n)$	Expected cost incurred by the structure for being in belief state ${}^av^n$ during time period n after application of action a;
$c(v^n)_{\|i,r,a^n}$	Expected cost at the beginning of time period n corresponding to belief state v^n given that we applied inspection technique i, got result r and implemented maintenance action a^n;
$*c(v^n)$	Optimal expected cost at the beginning of time period n corresponding to belief state v^n;
x_q	Strategy belonging to the set Δ_q of possible strategies for structure q (i.e. x_q is a set of decision rules which associates appropriate decisions for each state of the structure q during each of the time periods);
x	Global strategy belonging to the set Δ of possible global strategies (i.e. x is a Q-tuple $x = (x_1, \ldots, x_Q)$ where component x_q represents the strategy for structure q);
$Cr^n(x)$	Expected direct cost paid by the inventory manager for all the structures during time period n for IM&R decisions;
$Cr_q^n(x_q)$	Expected direct cost paid by the inventory manager for structure q during time period n for IM&R decisions;
$C(x)$	Overall expected long-term cost for all the structures which includes the discounted expected direct costs $Cr^n(x)$, in addition to the costs incurred by the user due to the states of the structures;
$C_q(x)$	Overall expected long-term cost for structure q which includes the discounted expected direct costs $Cr_q^n(x)$, in addition to the costs incurred by the user due to the state of structure q;
bl^n	Available budget for the inventory manager during time period n;
Q	Number of structures in the inventory;
α	Discount rate;
λ	Lagrange multiplier;
N	Time horizon of the planning.
L	Number of inspections performed at each stage

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Appendix

This appendix presents a procedure that we will refer to as 'procedure lambda' for the computation of the λ^n values.

(0) Set $\lambda^n = 0$ for $n = 1, \ldots, N$;
(1) Solve problem (7), calculate $Cr^n(x)$ for $n = 1, \ldots, N$;
(2) If $Cr^n(^*x) \approx bl^n$ for $n = 1, \ldots, N$ stop, else goto step 3;
(3) Add to every λ^n the value $\varepsilon = \frac{Cr^n(^*x) - bl^n}{bl^n} \times r$ (λ^n s are limited downward by 0);
(4) goto step 1;

r is a multiplying factor smaller than one.

More sophisticated procedures can be devised. However, for our particular problem the efficiency of the above listed procedure was found sufficient.